ON WRONSKIANS OF FUNCTIONS OF A REAL VARIABLE.

BY PROFESSOR MAXIME BÔCHER.

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THE important part played by the real roots of solutions of homogeneous linear differential equations of the second order is well known. When we pass to equations of higher order, it becomes necessary to consider not only the roots of the solutions of the equations, but also the roots of Wronskians* of sets of linearly independent solutions. Unless we are willing to confine ourselves to the case in which the coefficients of the differential equation, and therefore also the solutions, are analytic functions, a number of questions present themselves to us at the very outset—for instance the question whether such Wronskians can have an infinite number of roots in a given interval; and again the question to what extent the theory of the adjoint† differential equation remains valid when the coefficients of the differential equation are not assumed to be analytic but merely continuous functions. It is my object in the present paper to settle some of these questions and thus clear the way for further investigations. In doing this I have first considered the slightly more general subject of linear families of which the solutions of a homogeneous linear differential equation form a special case.

§ 1. *Exponents.*

We will say that a function‡ f of the real variable x has at a point c the exponent κ if throughout the neighborhood of c

$$f(x) = (x - c)^{\kappa} E(x) \tag{1}$$

* I shall denote the Wronskian of m functions $f_1, f_2, \cdots, f_m$, that is the determinant whose ith row is

$$f_1^{[i-1]} f_2^{[i-1]} \cdots f_m^{[i-1]},$$

by $W(f, \cdots, f_m)$.

† *Adjungirte.* Cf. Schlesinger, Handbuch, vol. 1, pp. 53–75, where proofs of the theorems in the case of analytic functions are given, as well as references to the original memoirs.

‡ The functions of a real variable used in this paper need not necessarily be real.

where $E(x)$ is continuous at c and throughout its neighborhood and $E(c) \neq 0$.

I. *A function cannot have two different exponents at the same point.*

Throughout the present paper we will denote by (J) the interval

$$(J) \qquad a \leqq x \leqq b.$$

II. *If $f(x)$ has an exponent at each point of (J) it cannot have more than a finite number of discontinuities in (J), nor can it have more than a finite number of roots in (J).*

For if $f(x)$ had an infinite number of discontinuities, or of roots, or of both, these points would have at least one limiting point c in (J). But since f has an exponent at c it is continuous and different from zero throughout the neighborhood of c.

We will from now on confine ourselves to the case of real exponents. If $f(x)$ has an exponent at c and if k is a constant not zero it is clear that $kf(x)$ has the same exponent there. If $f_1(x)$ and $f_2(x)$ have different real exponents at c, $f_1(x) + f_2(x)$ has an exponent at c, namely the smaller of the two exponents of f_1 and f_2. A combination of these two facts gives us the following theorem :

III. *If $f_1(x), f_2(x), \cdots f_m(x)$ have at c real exponents no two of which are equal, and if $k_1, k_2, \cdots, k_m$ are constants no one of which is zero, then the function*

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_m f_m(x)$$

has an exponent at c, namely the smallest of the exponents of $f_1, f_2, \cdots, f_m$.

By combining II. and III. we get the further theorem

IV. *If $f_1, f_2, \cdots, f_m$ have at c real exponents no two of which are equal, then these functions are linearly independent.**

If $f(x)$ is continuous at c, then to say that it has the exponent zero at c merely means that it does not vanish there ; while instead of saying that it has the positive integral exponent κ at c we might very well say that it has a κ-fold root at this point.† The following theorem shows that the familiar test for the multiplicity of a root is valid provided the necessary derivatives exist and are continuous :

* For a special case of this theorem cf. Heffter : Lineare Differentialgleichungen, p. 236.

† Thus for instance the property which κ-fold roots of real analytic funtions have that on opposite sides of such a root the function has the same or opposite signs according as κ is even or odd is at once seen to hold here if $f(x)$ is real.

V. *If throughout an interval including a point c $f(x)$ is continuous and has continuous derivatives of the first $\varkappa$ orders, a necessary and sufficient condition that $f(x)$ have the exponent $\varkappa$ at c is*

$$(2) \qquad f^{[i]}(c) = 0 \quad (i = 0, 1, \cdots, \varkappa - 1); \quad f^{[\varkappa]}(c) \neq 0.*$$

For if conditions (2) are fulfilled we have by Taylor's theorem, if we use Lagrange's formula for the remainder,

$$(3) \qquad f(x) = \frac{(x-c)^{\varkappa}}{\varkappa!} f^{[\varkappa]}[c + \theta(x-c)],\dagger$$

where θ is a function of x satisfying the inequality $0 < \theta < 1$. When $x \neq c$, $f^{[\varkappa]}[c + \theta(x-c)]$ is continuous because it is then equal to $\varkappa!(x-c)^{-\varkappa} f(x)$. When $x = c$, it is also continuous since $\lim_{x=c} f^{[\varkappa]}[c + \theta(x-c)] = f^{[\varkappa]}(c)$. Since this last quantity is not zero, (3) is a special case of (1) and therefore $f(x)$ has the exponent $\varkappa$.

Conversely, if $f(x)$ has the exponent $\varkappa$ at c, we see by the part of the theorem just proved that none of the quantities $f^{[i]}(c)$ $(i < \varkappa)$ can be different from zero. That $f^{[\varkappa]}(c) \neq 0$ follows at once from (3) when we let

$$f(x) = (x-c)^{\varkappa} E(x);$$

for we then find

$$(4) \qquad E(c) = \frac{1}{\varkappa!} f^{[\varkappa]}(c).$$

If we apply the theorem just proved and formula (4) not to $f(x)$ but to $f^{[j]}(x)$ we get the theorem ‡

VI. *If $f(x)$ has the exponent $\varkappa$ at c, and in an interval including c is continuous and has continuous derivatives of the first $\varkappa$ orders, then $f^{[j]}(x)$ $(j = 1, 2, \cdots, \varkappa)$ has the exponent $\varkappa - j$ at c, and if we write*

* If $f(x)$ has an exponent, not necessarily integral, greater than κ it can readily be proved that even though $f(x)$ has no derivatives of order higher than κ

$$f^{[i]}(c) = 0 \qquad (i = 0, 1, \cdots, \kappa).$$

† This formula must be slightly modified if $f(x)$ is a *complex* function of the real variable x, the real quantity θ being then different in the real and the imaginary part of $f[\varkappa]$. No difficulty, however, is introduced hereby; and in fact $f(x)$ may be complex throughout the whole of the present paper.

‡ It should be noticed that we cannot simply say that if $f(x)$ has the exponent κ, and if $f'(x)$ exists and is continuous, then $f'(x)$ has the exponent $\kappa - 1$. This is seen by considering the function $x^2 + x^3 \sin \frac{1}{x}$ at the point $x = 0$.

$$f(x) = (x-c)^{\kappa} E(x),\ f^{[j]}(x) = (x-c)^{\kappa-j} E_j(x),$$

then

$$E_j(c) = \kappa(\kappa-1)\cdots(\kappa-j+1)\,E(c). \tag{5}$$

§ 2. *Linear Families and Their Bases.*

Let $y_1, y_2, \cdots, y_k$ be functions of the real variable x which in the interval

$$a \leqq x \leqq b \tag{J}$$

are linearly independent. The family of functions represented by the formula

$$C_1 y_1 + C_2 y_2 + \cdots + C_k y_k, \tag{6}$$

the C's being arbitrary constants, we will call a *linear family.* By a *basis* of this family we understand a system of linearly independent functions $\eta_1, \eta_2, \cdots, \eta_m$ such that the linear family

$$K_1\eta_1 + K_2\eta_2 + \cdots + K_m\eta_m \tag{7}$$

is identical with (6). In particular therefore the y's form a basis. Moreover if the η's form a basis each of them must be a function of the family (6).

VII. *A necessary and sufficient condition that the functions* $\eta_1, \eta_2, \cdots, \eta_m$ *belonging to the family* (6) *form a basis is that they are linearly independent and that* $m = k$.

For expressing the η's as linear functions of the y's we easily see that if $m > k$ the η's must be linearly dependent, and therefore cannot form a basis.

On the other hand if $m < k$ and the η's formed a basis, the y's would form a basis of (7); and this is impossible by what we have just proved.

Finally if $m = k$, let us write

$$\eta_i = a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{ik}y_k \quad (i = 1, 2, \cdots k). \tag{8}$$

Substituting this in (7), we see that every function of (7) belongs to (6). If the η's are linearly independent the determinant of the a's must be different from zero. Equations (8) can therefore be solved for the y's, and these values substituted in (6). We thus see that every function of (6) belongs to (7); that is, (6) and (7) are identical.

Let us now impose on our linear family (6) the further restriction, that every function, except zero, of the family has a real exponent at the point c of (J).

Among all the functions of this family there cannot be more than k different exponents for a given point c of (J)

as we should otherwise, by IV., have more than k linearly independent functions in the family. It is less obvious that the number of distinct exponents at c cannot fall below k. This can, however, be proved as follows :

Suppose there were l and only l distinct exponents at c, and $l < k$. Denote these exponents in order of increasing magnitude by $\varkappa_1, \varkappa_2, \cdots, \varkappa_l$. Let $\eta_{i1}, \eta_{i2}, \cdots, \eta_{il}$ be functions which have at c the exponents $\varkappa_1, \varkappa_2 \cdots, \varkappa_l$ respectively. Since $\eta_{i1}, \eta_{i2}, \cdots, \eta_{il}$ do not form a basis, being less than k in number, there exist functions of the family which are linearly independent of them. Let $\varkappa_m$ be the largest of the exponents to which corresponds a function $\bar{\eta}_{im}$ of the family linearly independent of $\eta_{i1}, \eta_{i2}, \cdots, \eta_{il}$, and write

$$\eta_m = (x-c)^{\varkappa_m} E(x), \quad \bar{\eta}_m = (x-c)^{\varkappa_m} \bar{E}(x).$$

The function

$$\eta(x) = \eta_m(x) - \frac{E(c)}{\bar{E}(c)} \bar{\eta}_m(x),$$

may be written

$$\eta(x) = (x-c)^{\varkappa_m} \varphi(x),$$

where $\varphi(c) = 0$. It therefore does not have at c any of the exponents $\varkappa_1, \varkappa_2, \cdots, \varkappa_m$, and hence, since it is a member of the family, it must have an exponent larger than $\varkappa_m$ and must therefore be linearly dependent on the η's,

$$\eta_m(x) - \frac{E(c)}{\bar{E}(c)} \bar{\eta}_m(x) = c_1 \eta_1(x) + \cdots + c_l \eta_l(x).$$

But this makes $\bar{\eta}_m$ linearly dependent on $\eta_{i1}, \cdots, \eta_{il}$, and this is contrary to hypothesis. We have thus proved the theorem

VIII. *If in the linear family* (6) *every function except zero has at* c *a real exponent, then among these exponents there exist* k *and only* k *distinct values.*

We will speak of these k quantities as the exponents of the family at c.

Any set of k functions of the family corresponding at c to these distinct exponents will form a basis. Such a basis we will call a *principal basis* for the point c.

§ 3. *Roots of Wronskians.*

Let us now assume that $y_1, y_2, \ldots, y_k$, besides being linearly independent throughout (J), are continuous and have continuous derivatives of the first $n-1$ orders $(n \geqq k)$ throughout this interval. The same will then be true for every

function of (6), and we will assume that *no function of the family, except zero, vanishes together with its first $n-1$ derivatives at any point of (J).* Every function of the family will therefore, by V., have at any point of (J) an integral exponent z, where $0 \leqq z \leqq n-1$. We will now prove a similar theorem for the Wronskian of the y's*

IX. *At any point c of (J) the Wronskian*

$$W(y_1, y_2, \cdots, y_k)$$

has an integral exponent z, where

$$0 \leqq z \leqq k(n-k). \tag{9}$$

Let $z_1, z_2, \cdots, z_k$ be the exponents of the family at c, and let $\eta_1, \eta_2 \cdots, \eta_k$ be a principal basis of the family at this point. Then

$$W(\eta_1, \eta_2 \cdots, \eta_k) = CW(y_1, y_2, \cdots, y_k),$$

where C is a non-vanishing constant, namely the determinant of the transformation (8). Accordingly it will be sufficient to prove that $W(\eta_1, \cdots, \eta_k)$ has an exponent z at c satisfying the above written inequality.

Let us write

$$\eta_i(x) = A_i(x-c)^{\kappa_i}[1 + \varphi_i(x)] \qquad (i = 1, 2, \cdots, k), \tag{10}$$

where A_i is a non-vanishing constant and φ_i a function of x continuous throughout (J) and vanishing at c. We can write further

$$\eta_i^{(j)}(x) = A_i(x-c)^{\kappa_i - j}[z_i(z_i-1)\cdots(z_i-j+1) + \varphi_{ij}(x)] \tag{11}$$
$$(i = 1, 2, \cdots, k\,;\ j = 1, 2, \cdots, k-1),$$

where the functions φ_{ij} are continuous throughout (J) and vanish at c. This formula follows at once from VI. when $j \leqq z_i$, while when $j > z_i$ its correctness is seen directly when we let

$$\varphi_{ij}(x) = \frac{(x-c)^{j-\kappa_i}}{A_i} - \eta_i^{(j)}(x),$$

since the quantity $z_i(z_i-1)\cdots(z_i-j+1)$ is zero in this case.

Substituting (10) and (11) in $W(\eta_1, \cdots, \eta_k)$ we find

$$W(\eta_1, \cdots, \eta_k) = A_1 A_2 \cdots A_k (x-c)^{\kappa_1+\kappa_2+\cdots+\kappa_k - \frac{k(k-1)}{2}} \Delta(x), \tag{12}$$

where

* It may be noted that the derivatives of the y's do not necessarily have exponents at all points of (J). They may for instance vanish an infinite number of times in (J).

$$\Delta(x)=\begin{vmatrix} 1+\varphi_1(x) & \cdots & 1+\varphi_k(x) \\ x_1+\varphi_{11}(x) & \cdots & x_k+\varphi_{k1}(x) \\ x_1(x_1-1)+\varphi_{12}(x) & \cdots & x_k(x_k-1)+\varphi_{k2}(x) \\ \cdots\cdots & \cdots & \cdots\cdots \\ x_1(x_1-1)\cdots(x_1-k+2) & \cdots & x_k(x_k-1)\cdots(x_k-k+2) \\ \quad +\varphi_{1k-1}(x) & & \quad +\varphi_{kk-1}(x) \end{vmatrix}$$

Throughout (J) $\Delta(x)$ is continuous and, since all the φ's vanish at c, $\Delta(c)$ is equal to the product of the differences of the x's. No two of the x's being equal, $\Delta(c) \neq 0$.

Thus we see from (12) that $W(\eta_1, \cdots, \eta_k)$, and therefore also $W(y_1, \cdots, y_k)$, has at c the exponent

$$z = x_1 + x_2 + \cdots + x_k - \frac{k(k-1)}{2} \tag{13}$$

The least possible value of z is obtained by giving to x_1, $x_2, \cdots, x_k$ the values $0, 1, \cdots, k-1$; and the greatest possible value by giving them the values $n-k, n-k+1, \cdots, n-1$. Therefore

$$0 \leqq z \leqq k(n-k).$$

Combining the theorem just proved with II., we see that

X. *$W(y_1, \cdots, y_k)$ cannot vanish an infinite number of times in (J).**

When $W(y_1(c), y_2(c), \cdots, y_k(c)) \neq 0$ we have $z = 0$. Therefore

XI. *Except at the points where $W(y_1, \cdots, y_k) = 0$, the family (6) has the exponents $0, 1, \cdots, k-1$.*

Another consequence of (13), a direct proof of which is also very simple, is that, when $k = n$, $W(y_1, \cdots, y_n)$ vanishes nowhere in (J).†

Finally we introduce the conception of sub-families as follows:

Let $\eta_1, \eta_2, \cdots, \eta_m (m < k)$ be linearly independent functions of the family (6). With these functions as a basis we construct a family (7), which since it is wholly contained in (6) we call a sub-family of (6). These sub-families all satisfy the conditions stated at the beginning of this section,

* A special case of this is the theorem that $W(y_1, \cdots, y_k)$ is not identically zero. It should be noticed that we thus get a new proof of this theorem since we have made no use of it up to this point.

† Conversely it can readily be proved that if $y_1, \cdots, y_n$ are any functions which are continuous and have continuous derivatives of the first $n-1$ orders throughout (J) and whose Wronskian does not vanish in (J), then they form a basis of a linear family, none of whose members, except zero, vanishes together with its first $n-1$ derivatives at a point of (J).

and hence theorems IX., X., XI., apply to them. In particular we see that the Wronskian of any set of linearly independent functions of the family has an exponent at every point of (J).

§ 4. *The Adjoint Family.*

We will begin here by assuming merely that $y_1, \cdots, y_k$ are continuous and linearly independent throughout (J) and have throughout this interval finite derivatives of the first $k-2$ orders.

Let us consider the Wronskians

$$W_i = W(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_k) \quad (i = 1, 2, \cdots, k) \tag{14}$$

Let $\theta_1, \theta_2, \cdots, \theta_{k-1}$, be any $k-1$ functions of the family (6), say

$$\theta_i = a_{i1} y_1 + a_{i2} y_2 + \cdots + a_{ik} y_k \quad (i = 1, 2, \cdots, k-1). \tag{15}$$

It is at once seen that

$$W(\theta_1, \cdots, \theta_{k-1}) = c_1 W_1 + c_2 W_2 + \cdots + c_k W_k, \tag{16}$$

where

$$c_i = \begin{vmatrix} a_{11} & \cdots & a_{1i-1} & a_{1i+1} & \cdots & a_{1k} \\ a_{21} & \cdots & a_{2i-1} & a_{2i+1} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-11} & \cdots & a_{k-1i-1} & a_{k-1i+1} & \cdots & a_{k-1k} \end{vmatrix} \quad (i = 1, 2, \cdots, k). \tag{17}$$

Conversely if the constants c are given, the a's can be so determined that equations (17), and therefore also equations (16), hold.* That is,

XII. *The Wronskians of the functions of the family* (6) *taken* $k-1$ *at a time themselves form a linear family.*

Moreover the proof just given shows that either the functions $W_1, \cdots, W_k$ or, if they are linearly dependent, some functions taken from among them form a basis of this new family. In this last case we should have a relation of the form

$$c_1 W_1 + \cdots + c_k W_k = 0,$$

where the c's are not all zero. The a's being then computed

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* If all the c's are zero this is obvious If not, suppose $c_1 \neq 0$. We have then merely to choose all the a's except $a_{11}, a_{21}, \cdots, a_{k-11}$ so that when $i = 1$ (17) is satisfied, and then to determine the remaining a's by means of the equations

$$a_{i1} c_1 - a_{i2} c_2 + \cdots + (-1)^{k-1} a_{ik} c_k = 0 \quad (i = 1, 2, \cdots, k-i).$$

as above from (17), the θ's would be linearly independent, but $W(\theta_1, \cdots, \theta_{k-1}) = 0$. Thus we have the theorem

XIII. *The functions $W_1, \cdots, W_k$ are linearly independent unless there exist $k-1$ linearly independent functions in the family* (6) *whose Wronskian vanishes identically.**

Let us now impose on the functions $y_1, \cdots, y_k$ the further restrictions which are stated at the beginning of § 3. The k functions

$$z_i = (-1)^{k+i} \frac{W_i}{W(y_1, \cdots, y_k)} \qquad (i = 1, 2, \cdots, k) \tag{18}$$

we call† the functions *adjoint* to $y_1, \cdots, y_k$. We then have the theorem

XIV. *The functions formed by dividing the Wronskians of the functions of* (6) *taken $k-1$ at a time by the Wronskian of a basis of* (6) *themselves form a linear family—the family adjoint to* (6). *The functions adjoint to any basis of* (6) *form a basis of this adjoint family.*

To this may be added, if we refer to (13),

XV. *If $\eta_1, \cdots, \eta_k$ form a principal basis of* (6) *at the point c, then the adjoint functions $\zeta_1, \cdots, \zeta_k$ form a principal basis of the adjoint family at c. If the exponent of η_i at c is $\varkappa_i$, the exponent of ζ_i will be $k - 1 - \varkappa_i$.*

§ 5. *Applications to Differential Equations.*

We will now consider the differential equation

$$P(y) \equiv \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0, \tag{19}$$

where $p_1, \cdots, p_n$ are throughout (J) continuous functions of the real variable x. Theorems IX. and X. give us at once

XVI. *If $y_1, \cdots, y_k$ are linearly independent solutions of* (19), *and if c is any point of (J) then $W(y_1, \cdots, y_k)$ has an integral exponent $\varkappa$ at c, where*

$$0 \leqq \varkappa \leqq k(n-k).$$

XVII. *The Wronskian of k linearly independent solutions of* (19) *cannot vanish an infinite number of times in (J).*

Let us now look at the question of determining the multipliers of the equation (19), that is functions z such that

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* Sufficient conditions that this should not occur will be found in my paper in the *Transactions*, vol. 2, p. 139.

† Frobenius introduced this term in *Crelle*, vol. 77, p. 250.

$$(20) \qquad zP(y) = \frac{d}{dx}\left[q_1 \frac{d^{n-1}y}{dx^{n-1}} + q_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + q_n y\right].$$

It is clear that we must here have

$$z = q_1.$$

Equation (20) then becomes equivalent to the following set of equations:

$$(21) \qquad \begin{array}{llll} q_1' = p_1 q_1 - q_2, & q_2' = p_2 q_1 - q_3, \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ q_{n-1}' = p_{n-1} q_1 - q_n, & q_n' = p_n q_1. \end{array}$$

These form a set of simultaneous linear homogeneous differential equations of the first order for determining the q's. It is customary to eliminate $q_2, \cdots, q_n$ between equations (21) and thus get a homogeneous linear differential equation of the nth order—the equation adjoint to (19)—for q_1 (or z).* This method, however, is not open to us unless we are willing to assume that the coefficients, $p_1, p_2, \cdots, p_{n-1}$ have derivatives of orders $n-1, n-2, \cdots, 1$ respectively.

From equations (21) we infer directly that the multipliers of (19) form a linear family whose bases consist of n functions each. This family is in fact adjoint to the family of solutions of (19). This last fact is most readily proved by establishing by the method of Frobenius† the formula

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* We obtain of course in the same way equations for the other q's. The equation satisfied by q_i has been termed the $(n-i+1)$th adjoint equation, so that what is ordinarily called simply the adjoint equation would be the nth adjoint equation. Cf. Cels, *Ann. de l'Ec. norm. sup.*, 3d ser., vol. 8 (1891), p. 341, and some more recent papers in *Crelle*.

† Cf. *Crelle*, vol. 77 (1874) p. 248. The method consists of deducing (22) from a special case of the important general formula

$$(F) \qquad W(y_1, \cdots, y_k)[W(y_1, \cdots, y_m)]^{k-m-1} = W(w_{m1}, w_{m2}, \cdots, w_{m\,k-m}) \qquad (1 \leqq m \leqq k-1),$$

where

$$w_{mi} = W(y_1, \cdots, y_m, y_{m+i}) \qquad (i = 1, 2, \cdots, k-m).$$

Frobenius's method of proving the formula (F) consists in first establishing the formula directly when $m = 1$, and then using the method of mathematical induction. Each of these two steps requires a slight addition to make it rigorous when we are dealing with non-analytic functions of a real variable. Thus what we establish by the general step in the mathematical induction is

$$W(y_1, \cdots, y_k)[W(y_1, \cdots, y_m)]^{k-m-1}[W(y_1, \cdots, y_{m-1})]^{k-m} = W(w_{m1}, w_{m2}, \cdots, w_{m\,k-m})[W(y_1, \cdots, y_{m-1})]^{k-m},$$

$$(22)\quad z_i P(y) = \frac{d}{dx}\left[(-1)^{i-1}\frac{W(y\,y_1,\cdots,y_{i-1}.y_{i+1},\cdots,y_n)}{W(y_1,\cdots,y_n)}\right],$$

where $y_1,\cdots,y_n$ are any set of n linearly independent solutions of (19) and $z_1,\cdots,z_n$ are the functions adjoint to them. Thus we have proved the theorem

XVIII. *A necessary and sufficient condition that z be a multiplier of* (19) *is that it be a member of the linear family adjoint to the family which consists of the solutions of* (19).

GRUND IM HARZ,
July 20, 1901.

THE CONFIGURATIONS OF THE 27 LINES ON A CUBIC SURFACE AND THE 28 BITANGENTS TO A QUARTIC CURVE.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, August 20, 1901.)

Introduction.

AFTER determining * four systems of simple groups in an arbitrary domain of rationality which include the four systems of simple continuous groups of Lie, the writer was led to consider the analogous problem for the five isolated simple continuous groups of 14, 52, 78, 133, and 248 parameters. The groups of 78 and 133 parameters are related to certain interesting forms of the third and fourth degrees respectively.† They suggested the forms C (§ 1) and Q (§ 3).

It is shown in § 1 that the cubic form C defines the configuration of the 27 straight lines on a cubic surface in or-

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the functions $y_1,\cdots,y_k$ being supposed to be any functions of x which throughout (J) have continuous derivatives of the first $k-1$ orders. This establishes the truth of (F) at all points of (J) except where

$$W(y_1,\cdots,y_{m-1})=0.$$

If c is a point where this last equality holds two cases are possible: 1° there may be points in every neighborhood of c where the equality does not hold and where therefore (F) holds. In this case, on account of the continuity of both sides of (F), this formula holds also at c 2°, $W(y_1,\cdots,y_{m-1})$ may vanish identically throughout the neighborhood of c. In this case (F) also holds at c since all the Wronskians which occur in it vanish at c; cf. *Transactions*, vol. 2, p. 148.

* Abstract presented to the Society, Aug. 20, 1901, to appear in extenso in the *Transactions*. A note on the subject appeared in *Comptes rendus*, CXXXII. (1901), pp. 1547-8.

† Cartan, Thèses, Paris, 1894.

dinary space. After proving this result, the writer observed that the formulæ remained unaltered if the notation for the variables was chosen to be $x_i, y_i, z_{ij} \equiv z_{ji}$ $(i, j = 1, \cdots, 6;$ $j \neq i)$, a notation given by Burkhardt. * The notation (1) has been retained in view of the relation with the later sections and to retain uniformity with the notation of a paper † on the transformation group defined by the invariant C for an arbitrary domain of rationality.

The group of the configuration of the 27 lines on a cubic surface is exhibited in § 2. A study of the quartic form Q and the group of the configuration defined by it is made in §§ 3–6.

§ 1. *The 27 Lines on a Cubic Surface.*

A general cubic surface contains 27 straight lines such that

1°. Any one of the 27 lines A meets 10 other lines which intersect by pairs, forming with A 5 triangles. The total number of such triangles on the surface is therefore 45.

2°. Any two triangles ABC and $A'B'C'$ having no side in common determine uniquely a third triangle $A''B''C''$, such that the corresponding sides of the three triangles intersect and form three new triangles $AA'A''$, $BB'B''$. $CC'C''$. The former set of three triangles is said to constitute a *trieder*. Each triangle lies in exactly 16 trieders.

These two properties completely define the configuration of the 45 triangles formed by the 27 lines on the cubic surface.

We employ the 27 variables

$$x_i, y_i, z_{ij} \equiv -z_{ji} \quad (i, j = 1, \cdots, 6; j \neq i) \tag{1}$$

and consider the cubic form with 45 terms ‡

$$C \equiv \sum_{i \neq j}^{i,j=1,\ldots,6} x_i y_j z_{ij} + \sum z_{\lambda\mu} z_{\nu\rho} z_{\sigma\tau},$$

the second sum comprising the 15 terms of the Pfaffian $[1\,2\,3\,4\,5\,6]$, so that the subscripts have the following values:

12 34 56, 13 24 65, 14 23 56,
12 35 64, 13 25 46, 14 25 63,
12 36 45, 13 26 54, 14 26 35,

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* *Math. Annalen*, vol. 41, p. 339.

† Offered by the writer, July 18, 1901, to the *Quar. Jour. of Math.*

‡ Derived from the function J of Cartan (l. c., p. 143) upon replacing each y_j by $-y_j$. The character of J was not considered by Cartan.

15 23 64, 16 23 45,
15 24 36, 16 24 53,
15 26 43, 16 25 34.

Let the 27 lines on the cubic surface be represented by the variables (1), so that $z_{ji} \equiv -z_{ij}$ represents the same line that z_{ij} represents. We proceed to prove that *the arrangement of the* 27 *variables into the* 45 *triples given by the terms of* C *furnishes a suitable notation for the configuration of the* 45 *triangles formed by the* 27 *straight lines on a general cubic surface in ordinary space.*

Of the triples exhibited by the terms of C, exactly 5 contain x_i; exactly 5 contain y_i; exactly 5 contain z_{ij}; viz.,

$$x_{ij}y_jz_{ij}, \quad x_{ji}y_iz_{ji}, \quad z_{ij}z_{mn}z_{pq}, \quad z_{ij}z_{mp}z_{qn}, \quad z_{ij}z_{mq}z_{np},$$

where i, j, m, n, p, q form a permutation of 1, 2, 3, 4, 5, 6.

For i and j fixed, the triple $x_{ij}y_jz_{ij}$ lies in exactly 16 trieders

$$\begin{Bmatrix} x_i & y_j & z_{ij} \\ z_{kl} & x_k & y_l \\ y_k & z_{jk} & x_j \end{Bmatrix} \text{(four)}, \quad \begin{Bmatrix} x_i & y_j & z_{ij} \\ y_k & x_l & z_{lk} \\ z_{ik} & z_{lj} & z_{rs} \end{Bmatrix} \text{(twelve)},$$

where i, j, k, l, r, s form an even permutation of 1, 2, 3, 4, 5, 6.

The triple $z_{12}z_{34}z_{56}$ lies in 12 trieders of the type just given and in exactly four additional trieders

$$\begin{Bmatrix} 12 & 34 & 56 \\ 35 & 26 & 14 \\ 64 & 15 & 23 \end{Bmatrix}, \quad \begin{Bmatrix} 12 & 34 & 56 \\ 53 & 16 & 24 \\ 46 & 25 & 13 \end{Bmatrix}, \quad \begin{Bmatrix} 12 & 34 & 56 \\ 36 & 15 & 24 \\ 54 & 26 & 13 \end{Bmatrix}, \quad \begin{Bmatrix} 12 & 34 & 56 \\ 63 & 25 & 14 \\ 45 & 16 & 23 \end{Bmatrix}.$$

Similarly, $z_{ik}z_{ij}z_{rs}$ lies in exactly 16 trieders. Hence the notation is suitable to exhibit the configuration.

§ 2. *Group* G *of the Equation for the* 27 *Lines.*

The group G of the configuration of the 45 triangles formed by the 27 lines on a general cubic surfaces is composed of the literal substitutions on the variables (1) which leave the function C invariant. The determination of G has been effected by Jordan (Traité, pp. 316–329) and by the writer (Linear groups, Chapter XIV.). The notations of the latter treatment may be identified with the present notations as follows:

$$x_1 = R_0, \quad x_2 = R_{212}, \quad x_3 = R_{111}, \quad x_4 = R_{121}, \quad x_5 = R_{131}, \quad x_6 = R_{141},$$
$$y_1 = R_{112}, \quad y_2 = R_1, \quad y_3 = R_{210}, \quad y_4 = R_{120}, \quad y_5 = R_{130}, \quad y_6 = R_{140},$$
$$z_{12} = R_2, \quad z_{13} = R_{110}, \quad z_{14} = R_{220}, \quad z_{15} = R_{230}, \quad z_{16} = R_{240},$$
$$z_{23} = R_{211}, \quad z_{24} = R_{221}, \quad z_{25} = R_{231}, \quad z_{26} = R_{241}, \quad z_{34} = R_{222},$$
$$z_{35} = R_{232}, \quad z_{36} = R_{242}, \quad z_{45} = R_{142}, \quad z_{46} = R_{132}, \quad z_{56} = R_{122}.$$

It follows that G is generated by the substitutions

$$W: \quad (x_1y_2z_{12})(x_2y_3z_{23})(x_3y_1z_{13})(y_4x_4z_{56})(y_5x_5z_{46})(y_6x_6z_{45})(z_{14}z_{24}z_{34})(z_{15}z_{25}z_{35})(z_{16}z_{26}z_{36});$$

$$E_1: \quad (x_4y_5z_{26})(y_4y_1z_{13})(x_6z_{24}z_{25})(x_2z_{46}z_{56})(y_1z_{25}z_{34})(y_2z_{15}z_{14});$$

$$E_2: \quad (y_5y_6)(y_4z_{13})(y_3z_{14})(y_1z_{34})(x_4z_{56})(x_5z_{25})(x_6z_{26})(z_{15}z_{16})(z_{35}z_{36})(z_{45}z_{46});$$

$$E_3: \quad (y_2y_6)(y_4z_{13})(y_3z_{14})(y_1z_{34})(x_2z_{25})(x_5z_{16})(x_6z_{56})(z_{12}z_{16})(z_{23}z_{26})(z_{24}z_{46});$$

$$T: \quad (y_1y_6)(x_4x_6)(z_{14}z_{16})(z_{24}z_{26})(z_{34}z_{36})(z_{54}z_{56}).$$

Of these, W, E_1, E_2, and E_3 give rise to even substitutions on the 45 triangles, while T gives rise to an odd substitution. The group G of order 51840 therefore has a subgroup G_{25920} generated by W, E_1, E_2, E_3. A suitable product of the latter replaces x_1 by an arbitrary one of the 27 variables. Hence G_{25920} is transitive.

§ 3. *Character of the Quartic Form Q.*

We employ the 56 variables

$$(2) \qquad x_i,\ y_i,\ x_{ik} = -x_{ki},\ y_{ik} = -y_{ki} \quad (i, k = 1, \cdots, 7;\ k \neq i)$$

and consider the quartic form with 630 terms*

$$Q \equiv \sum x_iy_jx_{ik}y_{jk} + \sum x_{\lambda\mu}x_{\nu\rho}y_{\mu\nu}x_{\lambda\rho} + \sum (x_iy_{\lambda\mu}y_{\nu\rho}y_{\sigma\tau} + y_ix_{\lambda\mu}x_{\nu\rho}x_{\sigma\tau}),$$

where in the first sum $i, j, k = 1, \cdots, 7$; $i \neq j$, $i \neq k$, $j \neq k$; in the second sum λ, μ, ν, ρ run through the various permutations of $1, 2, \cdots, 7$ four at a time, but taking only one of the four equal terms

$$x_{\lambda\mu}x_{\nu\rho}y_{\mu\nu}y_{\lambda\rho} = x_{\mu\lambda}x_{\rho\nu}y_{\lambda\rho}y_{\mu\nu} = x_{\nu\rho}x_{\lambda\mu}y_{\rho\lambda}y_{\nu\mu} = x_{\rho\nu}x_{\mu\lambda}y_{\nu\mu}y_{\rho\lambda};$$

in the third sum $i, \lambda, \mu, \nu, \rho, \sigma, \tau$ denotes an even permutation of $1, 2, \cdots, 7$, so that the coefficient of x_i is the Pfaffian $[1\,2 \cdots i-1\ i+1 \cdots 7]$ defined in § 1.

Each of the three sums in Q extends over 210 terms. The fact that there are 630 terms in Q also follows from the lemma next proved, since $\frac{1}{4}\,45 \times 56 = 630$.

LEMMA.—*Each variable lies in exactly* 45 *terms of* Q. *The complementary cubic factor defines the configuration of the* 45 *triangles formed by the* 27 *straight lines on a cubic surface.*

* Suggested by the essentially different function J of Cartan (p. 144).

There are 30 and 15 terms respectively in the sums

$$\frac{\partial Q}{\partial x_i} \equiv \Sigma\, y_j x_{ik} y_{jk} + \Sigma\, y_{\lambda\mu} y_{\nu\rho} y_{\sigma\tau},$$

where $j, k = 1, \cdots, 7$; $j \neq i, k \neq i, j \neq k$; and $\lambda, \mu; \nu, \rho; \sigma, \tau$ run through the subscripts of the 15 terms of the Pfaffian $[1, 2, \cdots, i-1, i+1, \cdots, 7]$. To identify $\frac{\partial Q}{\partial x_i}$ with the function C of § 1, we have only to set $x_{ik} = -x_k$, $y_{jk} = z_{jk}$. From $\frac{\partial Q}{\partial x_i}$ we obtain $\frac{\partial Q}{\partial y_i}$ by interchanging y_j with x_j, y_{jk} with x_{jk}, an operation defining a substitution A which leaves Q unaltered. Similarly (§ 4) there exist substitutions which leave Q invariant and replace x_1 by any one of the 56 variables. Hence the lemma holds for each x_{ij} and y_{ij}. For a direct proof, consider the case

$$\frac{\partial Q}{\partial y_{67}} \equiv \sum_{i=1}^{5} (x_i y_6 x_{i7} + x_i y_7 x_{6i}) + \Sigma\, x_{6\mu} x_{\nu 7} y_{\mu\nu} + \Sigma\, x_i y_{\nu\rho} y_{\sigma\tau},$$

where in the second sum $\mu, \nu = 1, \cdots, 5$; $\mu \neq \nu$; in the third sum $i = 1, \cdots, 5$, while ν, ρ, σ, τ form a cyclic permutation of 1, 2, 3, 4, 5, with i excluded. To identify this expression of 45 terms with C, we may take

$$x_i = z_{6i},\ x_{i7} = y_i,\ x_{6i} = x_i,\ y_6 = x_6,\ y_7 = -y_6,\ y_{ij} = z_{ij}$$
$$(i, j = 1, \cdots, 5;\ j \neq i).$$

§ 4. *Substitution Group H with the Invariant Q.*

Among the literal substitutions on the 56 letters which leave Q invariant occur the following types:

$$A: \quad (x_i y_i)(x_{ij} y_{ij}) \qquad [i, j = 1, \cdots, 7;\ j \neq i];$$

$$B_{ij}: \quad (x_i y_j)(x_j y_i)(x_{ik} y_{kj})(x_{jk} y_{ki})(x_{ij} y_{ij})(x_k y_k)(x_{ik} y_{ik})$$
$$[k, l = 1, \cdots, 7, \text{ excluding } i, j];$$

$$D_1: \quad (x_1 y_1)(x_i x_{i1})(y_i y_{i1})(x_{ij} y_{jk}) \qquad [i, j, k = 2, \cdots, 6].$$

Each is the product of 28 transpositions. A suitable product of A, B_{ij}, D_1 throws x_1 to any one of the 56 letters.

The group H leaving Q invariant is, therefore, transitive.

The order Ω of H is therefore $56\,\Omega_1$, where Ω_1 is the order of the subgroup H_1 which leaves x_1 fixed. The substitutions of H_1 permute amongst themselves the 45 terms of $\frac{\partial Q}{\partial x_1}$. In

view of § 2, Ω_1 is *at most* equal to 51840 Ω', where Ω' is the order of the subgroup of H which leaves fixed the 28 variables x_1, y_j, x_{1j}, y_{jk} $(j, k = 2, \cdots, 7\,;\, j \neq k)$. We readily verify that the identity is the only substitution S of H which leaves these 28 variables fixed, so that $\Omega' = 1$ and therefore

$$\Omega \leqq 56 \times 51840. \tag{3}$$

In fact, S must permute amongst themselves the terms of Q which involve both y_7 and x_{1j} and hence leave invariant

$$\frac{\partial^2 Q}{\partial y_7 \partial x_{1j}} \equiv x_1 y_{7j} + x_j y_{17} + x_{\nu\rho} x_{\sigma\tau} + x_{\nu\sigma} x_{\tau\rho} + x_{\nu\tau} x_{\rho\sigma},$$

where $j, \nu, \rho, \sigma, \tau$ is an even permutation of 2, 3, 4, 5, 6. Hence the letters common to two such quadratic forms must be permuted amongst themselves. It follows that S leaves invariant

$$y_{17}, x_{46}, x_{56}, x_{45}, x_{24}, x_{25}, x_{26}, x_{23}, x_{34}, x_{35}, x_{36},$$

and therefore also x_2, x_3, x_4, x_5, x_6. Hence each term $x y_{1i} y_{\nu\rho} y_{\sigma\tau}$ is invariant and therefore x_7, y_{ij} $(j = 2, \cdots, 7)$. Similarly, $x_2 y_1 x_{23} y_{13}$ and $x_7 y_1 x_{7k} y_{1k}$ $(k = 2, \cdots, 6)$ require that y_1, x_{7k} be fixed. Hence S leaves all 56 letters fixed and is the identity.

A shorter proof follows from the important lemma

If a substitution of H leaves x_i fixed, it leaves y_i fixed, and inversely. If it leaves x_{ij} fixed, it leaves y_{ij} fixed and, inversely.

In view of the form of the substitutions A, B_{ij}, D_1 and the transitivity of H, it suffices to consider the case of a substitution Σ of H which leaves x_1 fixed. Then Σ permutes the 27 variables y_j, x_{1j}, y_{jk} $(j, k = 2, \cdots, 7)$ amongst themselves and therefore permutes the remaining 28 variables y_1 and

$$x_j, y_{1j}, x_{jk} \qquad (j, k = 2, \cdots, 7\,;\, j \neq k). \tag{4}$$

Hence Σ permutes the 45 terms of Q which involve y_1, these being the only terms containing exclusively letters of the set (4). Since each of the letters (4) occurs in terms of Q which do not involve y_1, no one of the letters (4) is replaced by y_1 under Σ. Hence y_1 is not altered by Σ.

In view of § 5, the group H_1 contains a subgroup holoedrically isomorphic with the group G_{51840}^{27} of the equation for the 27 lines on a cubic surface. It follows that

$$\Omega \leqq 56 \times 51840.$$

Combining this result with the result (3), we conclude that *the order of group H is* 56×51840.

§ 5. *Generators of the Group H_7.*

In investigating the subgroup H_7 of H which leaves x_7 fixed, we consider certain substitutions of H_7 which leave $\frac{\partial Q}{\partial x_7}$ fixed. By § 3, the latter function is identical with the cubic form C of § 1 if we let $x_{7k} = -x_k$, $y_{jk} = z_{jk}$, for $k, j = 1, \cdots, 6$; $k \neq j$. We obtain from the substitutions W, E_i, T of § 2 the following substitutions leaving $\frac{\partial Q}{\partial x_7}$ invariant:

$$[E_2]: \quad (y_3y_6)(y_4y_{13})(y_5y_{14})(y_1y_{34})(x_{77}y_{56})(x_{67}y_{23}) \\ (x_{67}y_{25})(y_{15}y_{16})(y_{35}y_{26})(y_{45}y_{46}),$$

$$[E_3]: \quad (y_2y_6)(y_4y_{13})(y_5y_{14})(y_1y_{34})(x_{77}y_{25})(x_{67}y_{26}) \\ (x_{67}y_{56})(y_{12}y_{16})(y_{23}y_{26})(y_{24}y_{46}),$$

$$[T]: \quad (y_4y_6)(x_{47}x_{67})(y_{14}y_{16})(y_{24}y_{26})(y_{34}y_{36})(y_{54}y_{56}),$$

together with $[W]$ and $[E_1]$, whose forms are equally evident. These substitutions do not leave Q invariant; but leave invariant $\frac{\partial Q}{\partial x_7}$ and $\frac{\partial Q}{\partial y_7}$. By interchanging x_i with y_i and x_{ij} with y_{ij} $(i, j = 1, \cdots, 7)$, we obtain the substitutions

$$A^{-1}[W]A,\ A^{-1}[E_i]A,\ A^{-1}[T]A \quad (i = 1, 2, 3)$$

affecting only the 27 variables x_j, y_{ij}. x_{ij} $(i, j = 1, \cdots, 6)$. Hence they are commutative with $[W]$, $[E_i]$, $[T]$. The products

$$(5) \qquad \begin{matrix} [W]A^{-1}[W]A,\ [E_i]A^{-1}[E_i]A, \quad (i = 1, 2, 3) \\ [T]A^{-1}[T]A \end{matrix}$$

therefore generate a group K which is holoedrically isomorphic with the group G^{27}_{51840} of the configuration defined by C, and hence with the group of the equation for the 27 lines on a cubic surface. Without proof,* I will state the theorem *that the substitutions* (5) *leave invariant the function* Q.

It follows from § 4 that the group K generated by the substitutions (5) is identical with the subgroup H_7 of the substitutions of H which leave x_7 invariant. Hence every substitution S of H_7 can be expressed as a product $S_1A^{-1}S_1A$, where S_1 affects only x_{ij}, y_j, y_{jk} $(j = 1, \cdots, 6)$, whereas $A^{-1}S_1A$ affects only y_{ij}, x_j, x_{jk}, being similar to S_1. To give a direct

* I first verified by direct computation that the theorem is true for $[E_3]A^{-1}[E_3]A$ and that $[E_2]A^{-1}[E_2]A$ is the transform of $[E_3]A^{-1}[E_3]A$ by B_{52}.

proof, let S replace y_j by $y_{\lambda\mu}$, for example. There exists a substitution $R = R_1 A^{-1} R_1 A$, where R_1 leaves x_j fixed and affects the same variables as does S_1, which replaces y_j by $y_{\lambda\mu}$ and therefore x_j by $x_{\lambda\mu}$. Then SR^{-1} leaves y_j fixed and therefore also x_j. Hence S replaces x_j by $x_{\lambda\mu}$.

§ 6. *Isomorphic Substitution Group on* 28 *Letters.*

THEOREM.—*The group H of all substitutions on the* 56 *variables* (2) *which leave Q invariant is imprimitive, possessing the* 28 *systems of imprimitivity*

$$(6) \qquad S_i \equiv [x_i, y_i], \quad S_{ij} \equiv [x_{ij}, y_{ij}] \qquad (i, j \quad 1, \cdots, 7;\ j \neq i).$$

Since A transforms each B_{ij} and D_1 into themselves, they preserve the imprimitive systems (6). We find that A corresponds to the identical substitution on S_i, S_{ij}; while

$$B_{ij} \sim (S_i S_j)(S_{ik} S_{kj}) \qquad (k = 1, \cdots, 7; k \neq i, j),$$
$$D_1 \sim (S_k S_{k1}) \qquad (k = 2, \cdots, 7),$$

each substitution on the systems being composed of 6 cycles. A suitable product of them replaces S_1 by any given one of the 28 systems. To the substitutions (5) correspond substitutions on the systems analogous to $[W]$, $[E_i]$, $[T]$, respectively. For example, to $[E_2]A^{-1}[E_2]A$ corresponds the substitution

$$(S_5 S_6)(S_4 S_{13})(S_3 S_{14})(S_1 S_{24})(S_{27} S_{56})(S_{57} S_{25})$$
$$(S_{67} S_{26})(S_{15} S_{16})(S_{23} S_{36})(S_{45} S_{46}).$$

The group H is hemiedrically isomorphic with a transitive substitution group H' of order 28×51840 *on the* 28 *letters S_i, S_{ij}.*

There is a known transitive substitution group of the same order and degree, viz., the group I' of the equation for the 28 bitangents to a quartic curve without double points. By the adjunction of a root,* the group reduces to a group holoedrically isomorphic with the group G of the 27 lines on a cubic surface (§ 2). By an earlier result, the subgroup of H' which leaves one letter fixed is holoedrically isomorphic with G. It would seem quite probable that I' and H' are isomorphic, and indeed identical groups. A formal investigation has not been attempted by the author. Granting the truth of this conjecture, the invariant Q would define the configuration of the 56 points of contact of the 28 bitangents to a quartic curve without double points.

THE UNIVERSITY OF CHICAGO,
June, 1901.

* Jordan, Traité des substitutions, p. 330.

THE FIFTIETH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE fiftieth annual meeting of the American Association for the Advancement of Science at Denver August 24–31, 1901, was the first meeting of this association held west of the Missouri River. There were 306 members and associates in attendance and about 220 papers were read before the various sections. Inaugurating a policy already announced, the Council voted that "each affiliated society is entitled to elect one member who is a fellow of the Association, as its representative in the Council; and if the society contains more than twenty five members who are fellows of the Association it is entitled to two representatives in the Council." The AMERICAN MATHEMATICAL SOCIETY would therefore be entitled to two representatives in the Council at its meetings with the Association, more than seventy of its members being also fellows of the Association.

The next meeting of the Association will be held at Pittsburg, June 28 to July 3, 1902, under the presidency of Professor Asaph Hall, of Harvard. Professor G. W. Hough, of Northwestern University, will be vice-president of Section A, and Professor E. S. Crawley, of the University of Pennsylvania, will be secretary. It was recommended that the next following meeting of the Association be held in Washington during the week including January 1, 1903.

The average attendance at the meetings of the section of mathematics and astronomy was about twenty-five. Most of the papers were followed by brief discussions. The officers of this section were: vice-president, James McMahon; secretary, G. A. Miller; councilor, G. B. Halsted; sectional committee, James McMahon, G. A. Miller, H. A. Howe, Florian Cajori, F. H. Loud; member of the general committee, C. A. Waldo. The Council elected the following mathematicians and astronomers to fellowship in the Association:

S. I. Bailey, E. M. Blake, C. L. Bouton, S. J. Brown, Wm. Cain, Florian Cajori, W. W. Campbell, C. W. Comstock, S. J. Cunningham, H. S. Davis, E. A. Engler, Irving Fisher, T. S. Fiske, G. B. Germann, S. Hart, H. A. Howe, C. J. Keyser, C. J. Ling, E. Miller, F. R. Moulton, H. B. Newson, G. A. Osborne, C. A. Post, T. J. J. See, W. B. Smith, J. M. Taylor, H. W. Tyler, J. H. Van Amringe.

On account of a change in the policy of the Association, the address of the vice-president of Section A will be delivered at the Pittsburg meeting. The following papers were presented at the meetings of this section:

(1) Professor G. B. HALSTED: "Supplementary report on non-euclidean geometry."

(2) Professor H. A. HOWE: "Kepler's problem for high planetary eccentricities."

(3) Professor H. A. HOWE and Miss L. L. STINGLEY: "The great fire ball of December 7, 1900."

(4) Professor FLORIAN CAJORI: "Conditionally convergent series whose product is absolutely convergent."

(5) Professor FLORIAN CAJORI: "On the application of the fundamental laws of algebra to infinite series."

(6) Professor L. E. DICKSON: "On systems of isothermal curves."

(7) Mr. C. J. KEYSER: "The plane geometry of the point in space of four dimensions."

(8) Professor H. A. HOWE and Miss M. C. TAYLOR: "The next opposition of Eros."

(9) Professor T. J. J. SEE: "On the probable densities of the satellites of the solar system."

(10) Mr. H. M. PARKHURST: "Photometric observations of Eros."

(11) Dr. G. A. MILLER: "On the history of several fundamental theorems in the theory of groups of finite order."

(12) Professor ARNOLD EMCH: "On certain methods in the geometry of position."

(13) Professor F. L. CHASE: "The parallex of 54 Piscium and of Weisse 17^h, 322."

(14) Professor F. L. CHASE: "The distance of the new star in Perseus."

(15) Mr. A. C. SMITH: "Hyperbolic curves of the nth order."

(16) Dr. F. R. MOULTON: "A uniform method of determining the elements of orbits of all eccentricities from three observations."

(17) Dr. J. I. HUTCHINSON: "On the modular functions associated with the irrationality

$$s^3 = z(z-1)\ (z-x)\ (z-y)."$$

(18) Professor F. H. LOUD and Mr. L. R. INGERSOLL: "Some future solar eclipses, in particular that of June 8, 1918, total at Denver."

(19) Professor ALEXANDER MACFARLANE: "Bibliography of quaternions and allied systems."

(20) Professor R. S. WOODWARD: "A summary of the salient effects due to the secular cooling of the earth."

(21) Professor R. S. WOODWARD: "The energy of condensation of stellar bodies."

(22) Professor C. J. LING: "The Bruce micrometer."

(23) Professor CLEVELAND ABBE: "The physical basis of long range weather forecasts."

(24) Mr. JESSE PAWLING, JR.: "The metric system in the United States."

(25) Mr. JESSE PAWLING, JR.: "The new bureau of standards."

Three of the titles which appeared on the preliminary programme are not reproduced here on account of the fact that these papers did not reach the secretary in time to be presented at the meeting. In the absence of the authors the papers by Professor See and Dr. Moulton were presented by Professor Howe. The papers by Professors Dickson and Macfarlane, Dr. Hutchinson, and Messrs. Keyser and Parkhurst were read by the secretary. In the absence of the member who was to present Professor Abbe's paper it was read by title. Abstracts of the papers, in order, are given below.

Professor Halsted's report was supplementary to the one read at the Columbus Meeting two years ago and printed in *Science*. It was mainly devoted to the advances made during the last two years. The advances contained in Dehn's article (published in the *Mathematische Annalen*, 1900) and in Hilbert's "Grundlagen der Geometrie," 1899, were especially emphasized. The report will appear in *Science*.

The solution of the equation $M = E - e \sin E$ is commonly called Kepler's problem. The known quantities e and M are respectively the eccentricity and the mean anomaly. E is to be found. The purpose of Professor Howe's first paper is to develop a direct method of solving Kepler's problem for planetary orbits of high eccentricity, which shall be more expeditious than any heretofore discovered, and shall be sufficiently accurate to meet the most exacting requirements of astronomers.

Professor Howe's second paper was devoted to the methods employed to collect data in reference to the great fire ball under consideration and to a discussion of its path. These two papers will appear in the *Astronomical Journal* and *Popular Astronomy* respectively.

In an article on "Divergent and conditionally convergent series whose product is absolutely convergent," in the *Transactions of the American Mathematical Society*, volume 2, 1901, p. 25, Professor Cajori gave special cases in which an absolutely convergent series is obtained as a result of multiplying two conditionally convergent series together or one conditionally convergent series by a divergent series. But the sum of one of the two factor series of each pair given in that article is zero. In the present paper it is shown that this is not a necessary property of conditionally convergent series whose product is absolutely convergent, and that the nth sum of such series may be of the degree $-r$ with respect to n, where $\frac{1}{2} < r \leqq 1$.

In his second paper Professor Cajori considers the behavior of infinite series with respect to the laws of algebra under two heads: An inquiry into the validity of the laws (1) when applied to the *terms* of an infinite series; (2) when applied to the infinite series *themselves*. The second inquiry, when made for the multiplication of series, leads to the conclusion that in this operation (assuming Cauchy's definition for the product of two infinite series), the associative, commutative and distributive laws are obeyed.

The two series obtained by removing the parentheses from the series

$$S_1 \equiv \sum_{p=0}^{p=\infty} \left(\frac{1}{4p^r+1} - \frac{1}{4p^r+4} + \frac{1}{4p^r+1} - \frac{1}{4p^r+4} \right),$$

$$S_2 \equiv \sum_{p=0}^{p=\infty} \left(\frac{1}{4p^r+4} + \frac{1}{4p^r+4} - \frac{1}{4p^r+1} - \frac{1}{4p^r+1} \right),$$

where $\frac{1}{2} < r \leqq 1$, are conditionally convergent, but their product is absolutely convergent. Hence, $(S_1S_2)(S_1S_2)(S_1S_2)$ is absolutely convergent. But $(S_1S_2)(S_1S_2)(S_1S_2) = S_1^3 \cdot S_2^3$, and S_1^3 and S_2^3 are each divergent when $r < \frac{2}{3}$. Hence, when $\frac{1}{2} < r < \frac{2}{3}$, S_1^3 and S_2^3 are two divergent series whose product is absolutely convergent.

The object of Professor Dickson's paper is to give an elementary geometrical definition of a system of sothermal curves in the plane. The definition is readily extended to families of curves on any algebraic surface. Two families of curves are discussed at length. From these the general definition is apparent. The paper has been published in the October number of the *American Mathematical Monthly*.

Mr. Keyser investigated the point (in 4-space) regarded as the assemblage of all the lineoids (*i. e.*, ordinary 3-fold spaces), planes, and lines containing it. This space is 3-dimensional in lineoids and in lines, the lineoid and the line being reciprocal elements; it is 4-dimensional in planes, the plane being *self-reciprocal*. The plane being taken as element, a theory results which is in its analytical aspect identical with the Plücker line theory of the lineoid, while the two theories are geometrically disparate. The subject was treated under the following six headings: Introductory considerations, concerning certain metric relationships, homogeneous coördinates of the plane, the linear complex of planes, linear congruences of planes, and projective transformations by means of complexes.

In his third paper, which will appear in *Popular Astronomy*, Professor Howe explained the computations which have been made for the rediscovery and observation of Eros when the conditions will become favorable. A chart giving the path of Eros through the sky was exhibited, and also a pasteboard model of the orbits of the earth and the planet, showing their positions at favorable oppositions.

Professor See points out the difficulty of measuring the angular diameters of very small bodies, on account of the tremors of the atmosphere, and then considers the densities of the great planets. He concludes that the average density of the four inner planets is 4.25, that of the outer planets 1.50, while that of the satellites is 2.36.

Mr. Parkhurst gave the results of observations extending from September 13th to March 22d and comprising 382 double extinctions in comparison with a large number of standard stars, including four other asteriods. The constant of brightness was ascertained to be 9.78 magnitude and the constant factor for phase angle .037.

Dr. Miller's historical account will appear in *Science*.

The paper by Professor Chase was supplementary to a paper read by him before this section a year ago under the title, "The series of parallaxes of large proper motion stars made with the Yale heliometer," a research begun in 1892, the observational part of which was finished the present year. In that paper the author had stated that the results of a preliminary solution indicated two of the 97 stars under

investigation to possess a parallax of nearly 0″.25, which values, if confirmed by further observation, would place them among the first ten or twelve nearest stars, so far as at present known. These two stars have been further investigated, two additional pairs of comparison stars being selected for each of them, and the observations with the original pairs repeated at the same time. Altogether there were 56 observations on 54 Piscium and 54 on Weisse 17^{h}, 322, distributed as follows:

54 Piscium, Mag. 6.2.

Series I.	12 obs. with a & b	Mags. 7.5 & 7.3	(orig.)
Series II.	12 obs. with a & b		(rep.)
Series III.	16 obs. with c & d	Mags. 8.7 & 8.7	
Series IV.	16 obs. with e & f	Mags. 7.5 & 5.5	

Weisse 17, 322 Mag. 8.0.

Series I.	10 obs. with a & b	Mags. 7.0 & 8.0	(orig.)
Series II.	12 obs. with a & b		(rep.)
Series III.	16 obs. with c & d	Mags. 7.2 & 5.5	
Series IV.	16 obs. with e & f	Mags. 8.6 & 7.2	

The observations treated in the customary way and the equations derived therefrom being solved, the following results were obtained:

for 54 Piscium from

Series I.	$\pi = +0''.241 \pm 0''.026$	Wt. 36.10
Series II.	$\pi = +0.081 \pm 0.017$	Wt. 42.26
Series III.	$\pi = +0.183 \pm 0.035$	Wt. 59.53
Series IV.	$\pi = +0.055 \pm 0.023$	Wt. 51.40

for Weisse 17, 322,

Series I.	$\pi = +0''.218 \pm 0''.030$	Wt. 33.60
Series II.	$\pi = +0.189 \pm 0.034$	Wt. 35.08
Series III.	$\pi = +0.198 \pm 0.022$	Wt. 45.87
Series IV.	$\pi = -0.047 \pm 0.031$	Wt. 38.92

When the new star in Perseus first appeared last February Professor Chase undertook to determine, if possible, the parallax of this most remarkable object. So far as is known to the author, no one has as yet succeeded in determining the distance of one of these new stars. In 1892 he began a series of observations on Nova Aurigæ for the same purpose, but it will be remembered that this star rapidly diminished in brilliancy, though with several fluctuations, and

was not observable with the Yale heliometer for more than two or three months, which would not give a very sensible parallax factor. With Nova Persei conditions have been much more favorable, and even now the star is conspicuously brighter than the brightest comparison star employed, which was, according to Argelander, of the 7.4 magnitude.

There was but a single pair of comparison stars suitable for the purpose, viz., B.D. + 43°, 720 Mag. 7.4 and B.D. + 43°, 766 Mag. 8.0. Calling the first *a* and the second *b*, the position angles were respectively about 252° and 94°, and the distances, 2900″ and 2700″ from the Nova.

The plan was to make the observations in the usual symmetrical order *Na*, *Nb*, *Nb*, *Na*, so as to eliminate, as far as possible, the effect of refraction and other effects which may vary with the time. Since the distance *ab* was not beyond the range of the heliometer it was thought expedient to measure this distance also each night, and thus have besides the sums of the distances an independent basis for correcting for changes in the scale value from night to night. Each night's work, then, consists of six observations of distance, each of four pointings in reversed positions of the instrument, as follows: *Na*, *Nb*, *ab*, *ab*, *Nb*, *Na*. A number of calculations were based upon these observations. Should the star remain sufficiently bright for another six months, it will then be possible to determine the effect of proper motion and hence give a definitive result.

In a plane are given n straight lines. Any point in the plane is taken as origin and through this a line is drawn, intersecting the n lines in n points. In this manner n segments measured from the origin, are obtained on the line through the origin. The algebraic sum of these segments taken on this same line will determine a point P which will describe a curve of the nth order as the ray through the origin rotates through 360°. Mr. Smith considered the locus of P and linear transformations of some of its special forms.

The methods of determining orbits which are in most general use were devised by Gauss one hundred years ago. They are different for orbits in which the eccentricities are less than, equal to, and greater than unity, although there is no singularity which is essential to the problem for the eccentricity equal to one. The method of Dr. Moulton's paper is uniform for all orbits, it is considerably more convenient than that of Gauss, and the radius of convergence of the series employed is examined in each case.

The longitude of the node and the inclination are computed by the usual methods, which are satisfactory, and the heliocentric distances and the arguments of the latitude at the epochs of the three observations are computed as in the method of Gauss.

Let u_1, u_2, u_3, r_1, r_2, r_3 represent the arguments of the latitude and the heliocentric distances at the epochs of the three observations. Then the parameter p is defined by the equation

$$k\sqrt{p}\,(t_3 - t_1) = \int_{u_1}^{u_3} r^2 du,$$

where r^2 is expressed as a series in u whose radius of convergence is determined as a function of u_2 and e. It is shown how the coefficients are to be found. The eccentricity e and the longitude ω of the perihelion from the node are given by

$$e \sin (u_1 - \omega) = \left\{ \frac{p - r_1}{r_1} \cos (u_3 - u_1) - \frac{p - r_3}{r_3} \right\} \operatorname{cosec} (u_3 - u_1),$$

$$e \cos (u_1 - \omega) = \frac{p - r_1}{r_1}.$$

The time of perihelion passage is determined from the law of areas.

The paper by Dr. Hutchinson aims to extend the results obtained by Picard in his memoir "Sur des fonctions de deux variables indépendantes analogues aux fonctions modulaires" (*Acta Mathematica*, volume 2, p. 114). Picard considers in the first place the integrals of the first kind and, in particular, the moduli of periodicity of the normal integrals. By changing the values of x, y in a continuous manner so as to return finally to their initial values, the moduli undergo a linear transformation, which can be represented by a linear transformation on two parameters u, v in terms of which all the moduli are rationally expressible. These transformations, forming an infinite group G, can be generated by five special ones S_1, S_2, ..., S_5, the explicit equations for which were given by Picard in a subsequent paper (*Acta Mathematica*, volume 5). The two variables x, y are then automorphic functions of u, v, and all functions belonging to the group can be rationally expressed in terms of these.

According to theorems previously obtained by Picard, there exist functions possessing a pseudo-automorphic character, exactly analogous to the fuchsian theta functions which Poincaré uses in connection with the automorphic functions of a single variable. These functions can be constructed out of the theta constants. In order to do this it is necessary to determine the effect of the transformations of the group G on the latter, which is accomplished by means of the transformation theory of the theta functions. A table is constructed by means of which pseudo-automorphic functions can readily be constructed.

The tables used in computing the circumstances of the eclipses discussed by Professor Loud are those "On the recurrence of solar eclipses" published by Professor Simon Newcomb in 1879. After some remarks upon the limits within which the errors of such a computation may be expected to fall, the results of what seems the preferable combination of Professor Newcomb's tabulated data are stated as follows:

On June 8, 1918, the moon's shadow passes across the United States from northwest to southeast, covering Denver from $4^h\ 22^m\ 59^s$ P. M. to $4^h\ 24^m\ 23^s$—a period of $1^m\ 24^s$; while on the central line the duration is $1^m\ 33^s$. The width of the shadow path is about 59.3^m and the velocity of the shadow $2{,}900^m$ an hour. The eclipse will be visible from Chamberlin Observatory, Denver, and Mt. Arapahoe, Grand Co.

Sept. 10, 1923. An eclipse of duration $3^m\ 24^s$ is total from San Diego, Cal., eastward along a line near the United States and Mexican boundary. Width of shadow path 102 miles.

Jan. 24, 1925. An eclipse visible (as total) from northern Michigan to New Haven, Ct., reaching the latter point at $9^h\ 5^m$ A. M., and lasting $2^m\ 8^s$.

Two maps of the Denver eclipse were shown.

The association for the promotion of quaternions and allied mathematics has in preparation a bibliography of all the literature of the subject. The field embraces all that has been written on what is called geometric algebra, or space analysis, and its three main subdivisions are quaternions, Ausdehnungslehre, and geometric algebra previous to Hamilton and Grassmann. As regards the first branch, the papers of Hamilton himself are numerous, and he has been followed by about one hundred writers. The writings

of Grassmann are also numerous, and he has been followed by about the same number of writers. It is desired to make the bibliography as complete as possible, and Professor Macfarlane requests mathematicians to coöperate by sending to him the necessary data about their own writings, or any rare writings in their possession.

The effects summarized in Professor Woodward's first paper are: (*a*) The slow process of heat conduction in the earth's crust, leading to the conclusion that nothing less than a million years is a suitable time unit for recording the historical succession of events. (*b*) The insignificant modifications of the process of conduction arising from the hydrosphere, leading to the conclusion that secular cooling goes on substantially as if the earth had neither atmosphere nor ocean. (*c*) The resultant effect on the lithosphere of secular contraction and the process of isostacy. (*d*) The effects of secular contraction on the length of the sidereal day.

The problem considered in Professor Woodward's second paper is that of the energy due to the gravitational condensation of gaseous matter from a state of infinite diffusion to a finite spherical mass in which Laplace's law of density holds. The problem is worked out in its generality, formulas specifying the distribution of density, pressure, and potential in the mass being given. Special attention is given to the probable case of the fixed stars of a vanishing surface density.

Mr. Ling exhibited and gave a description of the filar micrometer which the late Miss Caroline W. Bruce presented to Chamberlin Observatory.

The object of Professor Abbe's paper is to direct the attention of mathematical physicists toward meteorology and to set before the ordinary student of the daily weather map those fundamental mechanical and thermal principles that govern the phenomena of the atmosphere. The author believes that the present empirical rules and approximate general methods will eventually be replaced by precise methods of a mixed analytical, graphical, and numerical character, and that the time is now ripe for students of hydrodynamics and theoretical physics to apply themselves to the practical problems whose solution means so much to the progress of man in conquering nature. The author explained and stated many of the equations of motion and of condition and as to others he referred to special treatises. He indi-

cated the general character of the laws of resistance of continents, mountains, and oceans to the motion of the atmosphere as also the probable general characteristics of the desired solutions of the whole system of equations. His fundamental departures from all previous analytical studies consisted in the recognition of a so-called convectional friction or resistance due to the interference with each other of ascending and descending currents of air, and in the recognition of the division of the globe into land and water hemispheres having their poles at London and anti-London respectively.

Mr. Pawling's first paper gave a brief history of the action of Congress in regard to weights and measures. The author also spoke of the "English decimal association" and the good that a similar association might do in this country. In his second paper he gave an account of the new National bureau of standards, which is to be located in the suburbs of Washington. It has an appropriation of $300,000 and a laboratory costing about $200,000 is to be erected. The Bureau will employ a number of young men just graduated from universities giving them opportunities to develop along the lines which they wish to follow. It will also employ specialists to do work in their line. It standardizes three grades of weights and measures: 1° Those for commercial use, 2° those for manufacturing and technical processes and professions, 3° those for extreme accuracy for scientific purposes. Professor Nichols, of Cornell, added that the prospects seemed very good that the United States was about to take a place among the leading nations in scientific work along this line.

G. A. MILLER.

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RIEMANN-WEBER: PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS.

Die Partiellen Differentialgleichungen der Mathematischen Physik. Nach Riemann's Vorlesungen in vierter Auflage neu bearbeitet von HEINRICH WEBER, Professor der Mathematik an der Universität Strassburg. Braunschweig, Friedrich Vieweg und Sohn. Erster Band, 1900, xvii + 506 pp. Zweiter Band, 1901, xi + 527 pp.

RIEMANN'S lectures on the partial differential equations of mathematical physics and their application to heat conduc-

tion, elasticity, and hydrodynamics were published after his death by his former student. Hattendorff. Three editions appeared, the last in 1882; and few books have proved so useful to the student of theoretical physics. The object of Riemann's lectures was twofold: first, to formulate the differential equations which are based on the results of physical experiments or hypotheses; second, to integrate these equations and explain their limitations and their application to special cases. However, since Riemann's death, the development and extension of physics has been so great, especially in our knowledge of the properties of the luminiferous ether and of the properties of moving electrical charges—ions, and the advances in mathematical analysis have been so marked, that simply a revision of Hattendorff's treatise has become impossible. It has been felt for some years that what was most needed by students of physics was not a new edition of Riemann's lectures, but rather an entirely new book with the same purposes in view and with possibly the same general title. This has been given us at last by Professor Heinrich Weber, of Strassburg. The fact that Professor Weber still retains on the title-page of his two volumes the words, "nach Riemann's Vorlesungen" is certainly a tribute of the highest respect of one great scholar for another; for to Weber, and to him alone, belongs all the credit for this most important work.

These two volumes include all the subjects—and more—treated by Riemann in the two books which were prepared for publication by Hattendorff: the one already referred to on "Partielle Differentialgleichungen," and the other on "Schwere, Electricität, und Magnetismus." In the first volume of Weber are given: (1) a discussion of those analytical theorems and methods which are useful in a study of mathematical physics, (2) a brief presentation of the principles of geometry and mechanics, (3) a treatment of the mathematical theory of electricity and magnetism. The second volume contains: (1) selections from the theory of linear differential equations, (2) heat conduction, (3) theory of elasticity, (4) electrical vibrations, (5) hydrodynamics.

Some idea of the scope and usefulness of the work may be formed from a more detailed statement of the contents. In the section devoted to analytical theorems and methods there are eight chapters, viz: Definite integrals, Fourier's theorem, Infinite series, Fourier's series. Multiple integrals, Functions of a complex variable, Differential equations, and Bessel's functions. In the second section, that entitled

"Principles of geometry and mechanics," there are six most interesting chapters: Linear infinitesimal deformations, Vectors, Potentials, Illustrations of potentials, Spherical harmonics, and a brief discussion (Ueberblick) of the Principles of mechanics. The third and last section of volume I—Electricity and magnetism—has nine chapters: Electrostatics, Problems of electrostatics, Magnetism, Electro-kinetics, Electrolytic conduction, Steady electric currents, Flow of electricity in plates, Flow of electricity in space, and Electrolytic displacements.

The first section of volume II., that dealing with linear differential equations, includes four chapters: Integration by means of hypergeometrical series, Integration by means of definite integrals, Riemann's p-functions, Oscillation theorems. The portion on heat conduction has three chapters: The differential equation of heat conduction, Problems of heat conduction in one dimension, Heat conduction in a sphere. There are seven chapters given up to the theory of elasticity: General theory of elasticity, Statical problems of the theory of elasticity, Pressure on an elastic support, Motion of stretched strings, Riemann's method of integration, Vibrations of a membrane, General theory of the differential equation of a vibrating membrane. The section on electrical vibrations has three chapters: Electrical waves, Linear electrical currents, Reflection of electrical vibrations. In hydrodynamics there are six chapters: General principles, Motion of a rigid body in a fluid—hydrodynamic and mechanical portions, Discontinuous motion of fluids, Propagation of pulses in a gas, Air vibrations of finite amplitude.

Each of these chapters is brief and contains only the theorems and discussions which are necessary for the matter in hand; yet, until one reads each chapter and each paragraph, it is impossible to form an idea of the amount of information which is contained in these pages and of the enormous difficulty it must have been to condense the matter into such a small space and to select suitable illustrations. These last are, without exception, well chosen; and the references to recent literature are particularly noteworthy. Several of the chapters on electricity are specially interesting, as they deal with problems not ordinarily discussed in text-books. If it is possible to select from the many most excellent chapters of Professor Weber's book that one which seems the most noteworthy, that chosen would be the one entitled a "Brief discussion of the principles of mechanics." In this we have, in the short space of thirty-three pages,

the most satisfactory deduction and discussion of Hamilton's and Lagrange's equations that is known to the reviewer. Weber refers in this chapter several times to the important memoir by Hölder in the *Göttingen Nachrichten*, 1896, on the "Principles of Hamilton and Maupertius," which is by no means as well known as it should be. It would have been, however, so simple a matter to have added a few sentences in regard to dissipative forces and concealed motions that one regrets that Professor Weber exercised such self-restraint.

There is need of only a few words of what might be called criticism; for one reads the whole book with feelings of the keenest satisfaction. There are several cases, however, where the author has not used the simplest or most direct proof of certain theorems, *e. g.*, Stokes's theorem; yet, in defense, it must be acknowledged that the proof given in each case is one which was chosen in order to illustrate some important mathematical method. Again, in the discussion of differentiation under the integral sign, it would have been better to have included the case where the limits are functions of the parameter; and many other times the reader wishes for slight extensions of the treatment given. All English speaking students are so thoroughly accustomed to calling by the name "Green's theorem" the well known volume and surface integral transformation, or the identity that follows from it, that to have the name given to any other theorem of Green's, however important, seems wrong to them. In any case, it would have been well to have alluded to the Green's theorem just referred to, even if only as a special case of Gauss's theorem. As has been stated before, the excellence of the references to current literature is most marked; and one regrets the more the absence of some brief statement of the history of certain theorems; for one feels sure that Professor Weber's remarks on the subject would be so valuable. For instance, few mathematical theorems are more useful in physics than the statement of the possibility of the expression of any vector as the sum of a potential (or lamellar) vector and a solenoidal one; and a brief sketch of the history and application of this theorem would have been most interesting. A discussion of polarized distribution might have been added with advantage, as well as a more complete treatment of the energy relations of various distributions. Again, Fourier's method of approaching the problem of heat conduction might have been referred to, and the absence of Stokes's name in speaking of hydrodynamics seems unaccountable.

These remarks on what seem desirable additions to the volume of Professor Weber's are not to be regarded in the least as criticisms of the work; for there is no other book in any language which provides the student of physics with so much necessary imformation, so well selected and prepared. The only book which even attempts in part to do this is Professor Webster's treatise on "Electricity and magnetism," the first part of which is devoted to the general theorems of mathematics and mechanics. But while Webster's treatment on the whole is fuller and more satisfactory in some respects than the corresponding chapters in the volumes before us, it is by no means so extended. In the former, the physical facts and ideas are brought to the front; while in the latter emphasis is laid upon the limitations imposed by mathematical analysis.

In conclusion, the utmost satisfaction must be expressed with the paper, type, and cuts of the book. It is an unusual pleasure to have in one's hands a volume which has been rendered so attractive and intelligible by the efforts of the printer and publisher.

J. S. AMES.

NOTES.

THE closing (October) number of Volume II. of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Geometry of a simultaneous system of two linear homogeneous differential equations of the second order," by E. J. WILCZYNSKI; "Theory of linear groups in an arbitrary field," by L. E. DICKSON; "On certain aggregates of determinant minors," by W. H. METZLER; "Ueber die Anwendung der Cauchy'schen Multiplicationsregel auf bedingt convergente oder divergente Reihen," by A. PRINGSHEIM; "Ueber den Goursat'schen Beweis des Cauchy'schen Integralsatzes," by A. PRINGSHEIM; "New proof of a theorem of Osgood's in the calculus of variations," by O. BOLZA; "On certain pairs of transcendental functions whose roots separate each other," by M. BÔCHER; "On the system of a binary cubic and a quadratic and the reduction of hyperelliptic integrals of genus two to elliptic integrals by a transformation of the fourth order," by J. H. MCDONALD; "On the theory of improper definite integrals," by E. H. MOORE; "On the convergence and character of a certain form of continued fraction," by E. B. VAN VLECK; Notes and errata, volumes 1 and 2.

THE October number (volume 23, number 4) of the *American Journal of Mathematics* contains the following articles: "Memoir on the algebra of symbolic logic," by A. N. WHITEHEAD; "Secular perturbations of the planets," by G. W. HILL; "Representation of linear groups as transitive substitution groups," by LEONARD EUGENE DICKSON; "A class of number-systems in six units," by G. P. STARKWEATHER.

THE October number (second series, volume 3, number 1) of the *Annals of Mathematics* contains the following papers: "On the convergence of the continued fraction of Gauss and other continued fractions," by E. B. VAN VLECK; "On the differentiation of an infinite series term by term," by M. B. PORTER; "A note on geodesic circles," by J. K. WHITTEMORE; "Note on the functions defined by infinite series whose terms are analytic functions of a complex variable; with corresponding theorems for definite integrals," by W. F. OSGOOD; "Nim, a game with a complete mathematical theory," by C. L. BOUTON; "On the groups generated by two operators of order three whose product is also of order three," by G. A. MILLER; "On the invariants of a quadrangle under the largest subgroup, having a fixed point, of the general projective group in the plane," by W. A. GRANVILLE.

THE use of the centesimal division of the quadrant of a circumference has been authorized at the examinations for admission to the Ecole polytechnique at Paris. Beginning with 1902 tables in either sexagesimal or centesimal division may be used in trigonometrical computations, but beginning with 1905 the use of the latter will be obligatory with permission to check results by sexagesimal tables.

THE *Bibliotheca Mathematica* for September, 1901, contains a bio-bibliography, prepared by Professor G. ENESTRÖM, of mathematicians whose death occurred in the period 1881–1900, with references to published obituary notices.

PROFESSOR E. WÖLFFING, of Stuttgart is preparing a catalogue of non-periodical literature in mathematics and mechanics, soon to be ready for publication. It will contain about sixteen thousand titles arrange d under four hundred headings.

CAMBRIDGE UNIVERSITY.—Advanced mathematical courses for the current academic year are announced as follows:—

Michaelmas term, 1091.—By Professor Sir G. G. STOKES: Hydrodynamics, including viscosity, three hours.—By Pro-

fessor A. R. FORSYTH: Linear differential equations, three hours; Calculus of variations, two hours.—By Professor G. H. DARWIN: Figure of earth and precession, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electrodynamics, three hours.—By Mr. H. F. BAKER: Theory of functions (conformal representation, hypergeometric functions, discontinuous groups), three hours.—By Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Plane analytic geometry, three hours.—By Mr. G. T. WALKER: The electromagnetic field, three hours.—By Mr. B. A. W. RUSSELL: The principles of mathematics, two hours.—By Mr. J. H. GRACE: Invariants and geometrical applications, three hours.

Lent term, 1902.—By Professor Sir G. G. STOKES: Physical optics, three hours.—By Professor A. R. FORSYTH: Linear differential equations (continued) and automorphic functions, three hours.—By Professor G. H. DARWIN: Potential and attractions, three hours.—By Professor Sir R. S. BALL: Planetary theory, three hours.—By Dr. E. W. HOBSON: Fourier's series and functions of a real variable, three hours.—By Mr. J. LARMOR: Electrodynamics, with optical and thermodynamic applications, three hours.—By Mr. H. F. BAKER: Theory of equations (Galois theory).—By Mr. H. M. MACDONALD: Hydrodynamics, three hours.—By Mr. A. BERRY: Elliptic functions.—By Mr. G. T. WALKER: The electromagnetic theory of light, three hours.—By Mr. G. T. BENNETT: Linear and quadratic complexes, three hours.—By Mr. B. A. W. RUSSELL: The principles of mathematics (continued).—By Mr. I' A. BROMWICH: Matrices, quadratic and bilinear forms.—By Mr. E. T. WHITTAKER: The problem of three bodies, three hours.

Easter term, 1902.—By Professor Sir R. S. BALL: Cometary orbits, three hours.—By Mr. H. F. BAKER: Continuous groups.—By Mr. W. L. MOLLISON: Theory of potential and electrostatics, three hours.—By Mr. A. N. WHITEHEAD: Non-euclidean geometry.

Long vacation, 1902.—By Professor Sir R. S. BALL: Applications of modern geometry to dynamics, three hours (short course).

THE HON. B. A. W. RUSSELL will deliver in Cambridge University (England) a course of lectures on the Principles of Mathematics and Symbolic Logic, during the Michaelmas and Lent Terms. The lectures will deal with Peano's logic, the logic of relations, cardinal arithmetic, ordinal

arithmetic (including irrationals and continuity), and geometry (projective, descriptive, and metrical).

MR. W. BURNSIDE and Mr. J. GREAVES are the moderators, and Mr. J. G. LEATHAM and Mr. J. H. GRACE are the examiners for Part I. of the Cambridge mathematical tripos held in the spring of 1902.

DURING the recent summer quarter of the University of Chicago there were offered fourteen mathematical courses with a total registration of three hundred and seventeen.

AT the recent bicentennial celebration of the founding of Yale University, the degree of doctor of laws was conferred on Professor J. HADAMARD, who was present as the official representative of the University of Paris.

PROFESSOR W. H. METZLER, of Syracuse University, has been elected a corresponding member of the Royal society of Canada.

DR. E. NEUMANN, docent in applied mathematics and physics in the University of Halle, has been appointed to an associate professorship in the University of Breslau.

PROFESSOR MAX WOLF, of Heidelberg, has received a call to the University of Göttingen as professor of astronomy and director of the observatory.

DR. G. ROST has qualified as docent in mathematics at the University of Würzburg.

MR. FRANK B. LITTELL, of the U. S. Naval Observatory, has been appointed a professor of mathematics in the navy.

AT Harvard University the leave of absence of Professor B. O. PEIRCE has been extended, and Professor M. BÔCHER has returned after a year's leave of absence. Dr. E. V. HUNTINGTON has been appointed instructor in mathematics in the Lawrence Scientific School.

PROFESSOR WILHELM SCHELL has retired from the active duties of the professorship of theoretical mechanics at the Technische Hochschule of Karlsruhe after fifty-one years of service in the institution.

THE deaths are announced of P. M. POKROVSKY, professor of mathematics at the University of Kiew, of A. F. BERGER, docent in mathematics at the University of Upsala, and of F. CASPARY, a mathematician of the firm of Siemens and Halske, Berlin.

NEW PUBLICATIONS.

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BURKHARDT (H.). See JAHRESBERICHT.

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HANCOCK (H.). Mémoire sur les systèmes modulaires de Kronecker. (Thèse.) Paris, Gauthier-Villars, 1901. 4to. 119 pp.

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JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Unter besonderer Mitwirkung von F. Müller und A. Wangerin herausgegeben von E. Lampe und G. Wallenberg. Vol. 30: Jahrgang 1899. (In 3 Heften.) Heft 1. Berlin, Reimer, 1901. 8vo. 432 pp. M. 13.60

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MEHMKE (R.). See JAHRESBERICHT.

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MIORINI (W. v.). Ein Beitrag zur Centralprojektion der Kegelschnittlinien. (Progr.) Wien, 1901. 8vo. 9 pp.

MÜLLER (F.). See JAHRBUCH.

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VIVANTI (G.). Teoria delle funzioni analitiche. Milano, Hoepli, 1901. 16mo. 8 + 431 pp. (Manuali Hoepli.)

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GAJDECZKA (J.). Uebungsbuch zur Arithmetik und Algebra für die oberen Klassen der Mittelschulen. 4te Auflage. Wien und Prag, Tempsky, 1901. 8vo. 3 + 224 pp. Cloth. M. 3.00

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LOOMIS (E. S.). Original investigation; or, how to attack an exercise in geometry; with many model solutions and a complete discussion of the principles underlying the same. Boston, Ginn, 1901. 12mo. 6 + 63 pp. Cloth. $0.35

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ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Herausgegeben im Auftrage der Akademien der Wissenschaften zu München und Wien und der Gesellschaft der Wissenschaften zu Göttingen, sowie unter Mitwirkung zahlreicher Fachgenossen. (In 7 Bänden.) Vol. IV: Mechanik, redigiert von F. Klein. (In 2 Teilen.) Teil 1. Heft 1: A. Voss, Die Prinzipien der rationellen Mechanik. Leipzig, Teubner, 1901. 8vo. 4 + 121 pp.

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